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RATIO AND INVERSE MOMENTS OF MARSHALL-OLKIN EXTENDED BURR TYPE XII DISTRIBUTION BASED ON LOWER GENERALIZED ORDER STATISTICS

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Abstract: For the Marshall-Olkin extended Burr type XII distribution, we have developed some new explicit formulas for ratio and inverse moments of lower generalized order statistics in this brief note. The connection for moments of ordinary order statistics, record statistics, and other ordered random variable approaches may be developed using these explicit formulations. Additionally, the conditional moment of the lower generalized order statistics has been used to explore a characterization conclusion of this distribution.

Keywords: Lower generalized order statistics, order statistics, record values, Marshall-Olkin extended Burr type XII distribution, ratio and inverse moments, characterization.

Introduction

Kamps (1995) developed the idea of generalized order statistics (gos) as a broad foundation for ordered random variable models. Furthermore, a number of different models of ordered random variables are recognized as specific instances of gos, including order statistics, k-th upper record values, upper record values, increasingly Type II censoring order statistics, Pfeifer records, and sequential order statistics. These models are useful, for example, in dependability theory. Nevertheless, this framework cannot include decreasingly ordered random variables. As a result, this model is unsuitable for research, such as models with smaller record values and reversed sorted order statistics. In order to facilitate a common approach to descending order statistics, Pawlas and Szynal (2001) presented the idea of lower generalized order statistics (lgos) using the notion of gos. Burkschat et al. (2003) further examined this idea under the term dual generalized order statistics. Through a common method below, the lgos models allow us to investigate decreasingly ordered random variables such as reversed order statistics, lower record values, and lower Pfeirfer records.:

Suppose $X(1,n,m,k),\ldots,X(n,n,m,k)$, $(k \geq 1,\ m$ is a real number), are n *lgos* from an absolutely continuous cumulative distribution function $(cdf)\ F(x)$ with probability density function $(pdf)\ f(x)$, if their joint pdf is of the form

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[F(x_i)]^{m} f(x_i)\right)[F(x_n)]^{k-1} f(x_n) \qquad (1.1)$$

$$\left(\prod_{j=1}^{\quad}\right)\left(\prod_{i=1}^{\quad}\right)$$

on the cone $F^{-1}(1) > x_1 \geq x_2 \geq \ldots \geq x_n > F^{-1}(0)$,

where $\gamma_j = k + (n-j)(m+1) > 0$ for all j, $1 \leq j \leq n$, k is a positive integer and $m \geq -1$.

If $m = 0$ and $k = 1$, then this model reduces to the $(n-r+1)-$th order statistic, from the sample $X_1, X_2, \ldots, X_n$ and (1.1) will be the joint *pdf* of n order statistics. If $k = 1$ and $m = -1$, then (1.1) will be the joint *pdf* of the first n record values of the identically and independently distributed (*iid*) random variables with *cdf* $F(x)$ and corresponding *pdf* $f(x)$.

In view of (1.1), the marginal *pdf* of the $r-$th *lgos*, $X^*(r,n,m,k)$, $1 \leq r \leq n$, is

$$f_{X^*(r,n,m,k)}(x) = C_{r-1}\frac{[F(x)]^{\gamma_r - 1} f(x)\, g_m^{r-1}(F(x))}{(r-1)!}, \qquad (1.2)$$

and the joint *pdf* of $X^*(r,n,m,k)$ and $X^*(s,n,m,k)$, $1 \leq r < s \leq n$, is

$$f_{X^*(r,n,m,k),X^*(s,n,m,k)}(x,y) = C_{s-1}\frac{[F(x)]^m f(x) g^{r-1}(F(x))}{(r-1)!(s-r-1)!}$$

$$\times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s - 1} f(y), \quad x > y, \qquad (1.3)$$

where

$$C_{r-1} = \prod_{i=1}^{r}\gamma_i, \quad h_m(x) = \begin{cases} -\frac{1}{m+1}x^{m+1}, & m \neq -1 \\ -\ln x, & m = -1 \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(1), \quad x \in [0,1).$$

Moments of order statistics play an important role in quality control testing and reliability to predict the failure of future items based on the times of few early failures. An application of the first moments of order statistics can be considered in calculating the L-moments which are in fact the linear combinations of the expected order statistics. See Hosking (1990) for details.The concept of recurrence relations has its own importance. They are useful in reducing the number of operations necessary to obtain a general form for the function under consideration. Furthermore, they are used in characterizing the distributions, which is an important area, permitting the identification of population distribution from the properties of the sample. Recurrence relations and identities have attained importance as it reduces the amount of direct computation, time and labour.

Various developments on *lgos* and related topics have been studied by Pawlas and Szynal (2001), Ahsanullah (2004, 2005), Mbah and Ahsanullah (2007). Recurrence relations for single and product moment of *lgos* from the exponentiated Kumarswamy, exponentiated Pareto, exponentiated gamma and generalized exponential distributions are derived by Kumar (2011) and Khan and Kumar (2010, 2011a, 2011b) respectively. Kumar (2013a, b) established recurrence relations for marginal and joint moments generating function of *lgos* from extended type I generalized logistic distribution and generalized logistic distribution respectively. Keseling (1999) characterized some continuous distributions based on conditional distributions of *gos*. Bieniek and Szynal (2003) characterized some distributions via linearity of regression of *gos* . Cramer *et al.* (2004) gave a unifying approach on characterization via linear regression of ordered random variables.

1.1. Marshall-Olkin extended Burr type XII distribution

Burr (1942) presented a family of distributions with twelve distinct kinds of cumulative distribution functions that might be beneficial for fitting data when modeling lifespan or survival data. Burr Type X and Burr Type XII were given the most attention out of the twelve distribution functions. These distributions have been widely employed in many different domains, including survival analysis, reliability theory, simulation, quantal response, approximation of distributions, and creation of non-normal control charts. Burr distributions are limiting variants of many common theoretical distributions. When analyzing lifespan and survival statistics, the Burr type XII distribution is crucial. In order to simulate severe events with regard to flood frequency, Shao et al. (2004) examined models for the extended three-parameter Burr type XII distribution. Rodriguez (1977) investigated the adaptability of the Burr type XII distribution. Numerous scientific disciplines, including actuarial science, forestry, ecoloxicology, dependability, and survival analysis, have made extensive use of the Burr type XII.

First presented by Marshall and Olkin (1997), the Marshall-Olkin extended Burr type XII distribution is rather easy to create. In order to create new families of distributions that are more adaptable and exhibit a greater variety of behavior than the original distributions, a parameter is added. This distribution's relevance in real-world scenarios is substantial since it is related to many distributions, including Burr type XII and the Marshall-Olkin extended Lomax distribution. Since then, a number of writers have created other Marshall-Olkin type distributions based on several well-known distributions, and the Marshall-Olkin distributions have been extensively explored in statistics. The Marshall-Olkin extended Pareto distribution with application by Ghitany (2005), the Marshall-Olkin extended Weibull distribution by Ghitany et al. (2005), the Marshall-Olkin extended Lomax distribution by Ghitany et al. (2007), the Marshall-Olkin Extended Uniform Distribution by Josea and Krishna (2011), the Marshall-Olkin Extended Log-Logistic Distribution by Gui (2013), and so forth.

We provide a thorough description ratio and inverse moments of lower generalized order statistics of this distribution in the hopes that they would draw broader applications in lifetime analysis, which makes this work unique.

Al-Saiari et al. (2014) presented the Marshall-Olkin extended Burr type XII distribution. He investigated a few characteristics and the calculation of the unknown parameters' greatest likelihood.A random variable X is said to have Marshall-Olkin extended Burr type XII distribution if its *pdf* is of the form

$$f(x) = \frac{\alpha\ \beta\ \lambda x^{\beta-1}(1+x^{\beta})^{-(\lambda+1)}}{[1-(1-\alpha)(1+x^{\beta})^{-\lambda}]^2}, \quad x>0,\ \alpha\ ,\ \beta,\lambda>0 \tag{1.4}$$

and the corresponding *cdf* is

$$F(x) = \frac{1-(1+x^{\beta})^{-\lambda}}{1-(1-\alpha)(1+x^{\beta})^{-\lambda}}, \quad x>0,\ \alpha\ ,\ \beta,\lambda>0 \tag{1.5}$$

and the survival function is given by

$$S(x) = \frac{\alpha(1+x^{\beta})^{-\lambda}}{1-(1-\alpha)(1+x^{\beta})^{-\lambda}}, \quad x>0,\ \alpha\ ,\ \beta,\lambda>0. \tag{1.6}$$

Burr XII and Marshall-Olkin extended Lomax distributions are consider as a special case of Marshall-Olkin extended Burr type XII distribution when $\alpha=1$ and $\beta=1$ respectively.

The paper is organized as follows. Some new expressions of the inverse moments of *lgos* from Marshall-Olkin extended Burr type XII distribution are derived in Section 2. In Section 3, some explicit expressions of ratio moments of *lgos* are derived. In Section 4, we obtained a characterization result of this distribution by using conditional expectation of *lgos*. Section 5 end with the some applications.

2. Relations for inverse moments

In this Section, the explicit expressions for inverse moments of *lgos* are considered. For the Marshall-Olkin extended Burr type XII distribution as given in (1.5), the $j-$th moments of $X^*(r,n,m,k)$ is given as

$$E[X^{*j-\beta}(r,n,m,k)] = \int_0^{\infty} x^{j-\beta} f_{X^*(r,n,m,k)}(x)\,dx$$

$$= \frac{C_{r-1}}{(r-1)!} \int_0^{\infty} x^{j-\beta}[F(x)]^{\gamma_r-1} f(x)\, g^{r-1}(F(x))\,dx \tag{2.1}$$

Further, on using the binomial expansion, we can rewrite (2.1) as

$$E[X^{*j}(r,n,m,k)] = \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u}$$

3. Relations for ratio moments

In this Section, the explicit expressions for ratio moments of *lgos* are considered. For Marshall-Olkin Burr type XII distribution, the ratio moments of $X^*(r,n,m,k)$ and

On using (1.3) and binomial expansion, we have

$$E[X^{*i}(r,n,m,k)\ X^{*j-\beta}(s,n,m,k)]$$

$$= \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)} \sum_{u=0}^{s-2} \sum_{v=0}^{r-1} (-1)^{u+v} \binom{s-2}{u} \binom{r-1}{v} \times \int_0^{\infty} x^i [F(x)]^{(s-r+u-v)(m+1)-1} f(x) I(x) dx, \quad x > y, \tag{3.1}$$

where

$$I(x) = \int_0^x y^{j-\beta} [F(y)]^{\gamma_{s-v}-1} f(y) dy. \tag{3.2}$$

By setting $t = F(y)$ in (3.2), we obtain

$$I(x) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^{l+(j/\beta)-1} (1-\alpha)^l (p/\lambda) \Gamma\left(1-\frac{j}{\beta}+p\right) \Gamma\left(\frac{j}{\beta}+q\right) \binom{p}{l} [F(x)]^{\gamma_{s-v}+q+l}}{p!q!l! \Gamma\left(1-\frac{j}{\beta}\right) \Gamma\left(\frac{j}{\beta}+1-l\right) [\gamma_{s-v}+q+l]}.$$

On substituting the above expression of $I(x)$ in (3.1), we find that

$$E[X^{*i}(r,n,m,k)\ X^{*j-\beta}(s,n,m,k)] = \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \times \sum_{u=0}^{r-1} \sum_{v=0}^{s-r-1} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^{u+v+l+(j/\beta)-1} (1-\alpha)^l (p/\lambda) \binom{r-1}{u} \binom{s-r-1}{v} \Gamma\left(1-\frac{j}{\beta}+p\right) \Gamma\left(\frac{j}{\beta}+q\right) \binom{p}{l}}{p!q!l! \Gamma\left(1-\frac{j}{\beta}\right) \lambda^{\gamma_{r-u}+q+l}} \times \int_0^{\infty} x^i [F(x)]^{\gamma_{r-u}+q+l-1} f(x) dx. \tag{3.3}$$

$$\times \frac{\Gamma\left(1-\frac{j}{\beta}+p\right)\Gamma\left(\frac{p}{\lambda}+q\right)}{\Gamma\left(1-\frac{j}{\beta}\right)\left(p+1-l\right)_{u=0}}{}_{r-1}\sum(-1)^{u}\binom{r-1}{u}\binom{k}{m+1}B\left|_{m+1}\left(\frac{q+l+a+t}{m+1}+n-r+u+1,\right.\right.$$

$$\times \frac{\Gamma\left(\frac{i}{\beta}+1\right)\Gamma\left(\frac{w}{\lambda}+a\right)}{\Gamma\left(\frac{i}{\beta}+1-w\right)\Gamma\left(\lambda+1-t\right)}\sum_{v=0}^{s-r-1}(-1)^{v}\binom{s-r-1}{v}\binom{k}{m+1}B\left|_{m+1}\left(\frac{q+l}{m+1}+n-s+v+1,\right.\right.\Big).$$

(3.4)

On using relation (2.4) in (3.4), we get

$$E[X^{*i}(r,n,m,k)\ X^{*j-\beta}(s,n,m,k)]=\frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}$$

$$\times\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{l=0}^{\infty}\sum_{a=0}^{\infty}\sum_{t=0}^{\infty}\sum_{w=0}^{\infty}\frac{(-1)^{l+w+t-1+(i+j)/\beta}(1-\alpha)^{l+t}\binom{p+w}{\lambda}}{p!q!l!w!t!a!}$$

$$\times\frac{\Gamma\left(1-\frac{j}{\beta}+p\right)\Gamma\left(\frac{p}{\lambda}+q\right)\Gamma\left(\frac{i}{\beta}+1\right)\Gamma\left(\frac{w}{\lambda}+a\right)}{\binom{j}{p}\binom{i}{w}\Gamma\left(1-\frac{j}{\beta}\right)\Gamma\left(\lambda+1-l\right)\Gamma\left(\frac{i}{\beta}+1-w\right)\Gamma\left(\lambda+1-t\right)}$$

$$\times B\left(\frac{k}{m+1}+n-r+\frac{q+l+a+t}{m+1},r\right)B\left(\frac{k}{m+1}+n-s+\frac{q+l}{m+1},s-r\right).$$

Which after simplification yields

$$E[X^{*i}(r,n,m,k)\ X^{*j-\beta}(s,n,m,k)]$$

$$=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{l=0}^{\infty}\sum_{a=0}^{\infty}\sum_{t=0}^{\infty}\sum_{w=0}^{\infty}(-1)^{l+w+t-1+(1+j)/\beta-1}(1-\alpha)^{l+t}\binom{p+w}{}$$

$$\times \prod_{\phi=1}^{r} \left(1+\frac{1}{\gamma_\phi}\right)^{s} \prod_{\psi=r+1}^{s} \left(1+\frac{q+l}{\gamma_\psi}\right). \qquad (3.5)$$

Special cases

i) Setting $m=0$, $k=1$ in (3.5), we get product moments of order statistics from Marshall-Olkin Burr type XII distribution as

$$E[X_{n-r+1:n}^{i} X_{n-s+1:n}^{j}] = \frac{n!}{(n-s)!} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{a=0}^{\infty}\sum_{t=0}^{l+t}\sum_{l=0}^{p+w}\sum_{w=0}^{i+p(j+q)/\beta} (-1)^{l+t+w-i+(i+j)/\beta}$$

$$\times \binom{p+w}{l+t}\binom{j}{p}\binom{i}{w}\frac{(1-\alpha)^{p}\,\Gamma\left(1-\frac{i+p}{\beta}\right)\Gamma\left(\lambda+q\right)\Gamma\left(\frac{j+1}{\beta}\right)\Gamma\left(\lambda+a\right)}{p!q!l!t!w!a!\,\Gamma\left(1-\frac{j}{\beta}\right)\Gamma\left(\lambda+1-l\right)\Gamma\left(\frac{i+1}{\beta}+1-w\right)\Gamma\left(\lambda\right)}$$

$$\times \frac{\Gamma(n-r+1+l+q+a+t)\,\Gamma(n-s+1+q+l)}{\Gamma(n+1+q+l+a+t)\,\Gamma(n-r+1+q+l)}.$$

Replace $n-s+1$ by r and $n-r+1$ by s

$$E[X_{r:n}^{i} X_{s:n}^{j}] = \frac{n!}{(r-1)!} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{a=0}^{\infty}\sum_{t=0}^{l+t}\sum_{l=0}^{p+w}\sum_{w=0}^{i+p(i+j)/\beta} (-1)^{l+t+w-1+(i+j)/\beta}$$

$$\times \binom{p+w}{l+t}\binom{j}{p}\binom{i}{w}\frac{(1-\alpha)^{p}\,\Gamma\left(1-\frac{i+p}{\beta}\right)\Gamma\left(\lambda+q\right)\Gamma\left(\frac{j+1}{\beta}\right)\Gamma\left(\lambda+a\right)}{p!q!l!t!w!a!\,\Gamma\left(1-\frac{j}{\beta}\right)\Gamma\left(\lambda+1-l\right)\Gamma\left(\frac{i+1}{\beta}+1-w\right)\Gamma\left(\lambda\right)}$$

$$\times \frac{\Gamma(s+l+q+a+t)\,\Gamma(r+q+l)}{\Gamma(n+1+q+l+a+t)\,\Gamma(s+q+l)}.$$

ii) If $m=-1$ in (3.5), we get product moments of $k-$th record values from Marshall-Olkin Burr type XII distribution as

4. Characterization

Let $X^*(r,n,m,k)$, $r=1,2,\ldots,n$ be *gos* from a continuous population with *df* $F(x)$ and *pdf* $f(x)$, then the conditional *pdf* of $X^*(s,n,m,k)$ given $X^*(r,n,m,k)=x$, $1\le r<s\le n$, in view of (1.3) and (1.2), is

$$f_{X^*(s,n,m,k)|X^*(r,n,m,k)}(y\mid x)=\frac{C_{s-1}}{(s-r-1)!C_{r-1}}$$

$$\times\frac{[(h_m(F(y))-h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s-1}}{[F(x)]^{\gamma_{r+1}}}f(y),\ x<y. \qquad (4.1)$$

Theorem 4.1: Let X be a non negative random variable having an absolutely continuous distribution function $F(x)$ with $F(0)=0$ and $0<F(x)<1$ for all $x>0$, then

$$E[X^*(s,n,m,k)\mid X^*(r,n,m,k)=x]=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{p+q+(1/\beta)}(1-\alpha)^q(p/\lambda)}{p!q!l!}$$

$$\times\left[\frac{1}{(1+x^{\beta})^{\lambda}-1}\right]^{q+l}\frac{\Gamma\left(\frac{1}{\beta}+1\right)\Gamma\left(\frac{p}{\lambda}+l\right)}{\Gamma\left(\frac{1}{\beta}+1-p\right)\Gamma\left(\frac{p}{\lambda}+1-q\right)}\prod_{j=1}^{s-r}\frac{\gamma_{r+j}}{\gamma_{r+j}+q+l}, \qquad (4.2)$$

if and only if

$$F(x)=\frac{1-(1+x^{\beta})^{-\lambda}}{1-(1-\alpha)(1+x^{\beta})^{-\lambda}},\qquad x>0,\ \alpha,\ \beta,\lambda>0.$$

Proof: From (4.1), we have

$$E[X^*(s,n,m,k)\mid X^*(r,n,m,k)=x]=\frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}$$

$$\left[(F(y))^{m+1}\right]^{s-r-1}(F(y))^{\gamma_s-1}\qquad f(y)$$

from (1.5) in (4.3), we obtain

$$\times \frac{\Gamma\left(\frac{1}{\beta}+1\right)\Gamma\left(\frac{p}{\lambda}+l\right)}{\Gamma\left(\frac{1}{\beta}+1-p\right)\Gamma\left(\frac{p}{\lambda}+1-q\right)} \int_0^1 u^{\gamma_s+q+l-1}(1-u^{m+1})^{s-r-1}du . \qquad (4.4)$$

Again by setting $t = u^{m+1}$ in (4.4) and simplifying the resulting expression, we derive the relation given in (4.2).

To prove sufficient part, we have from (4.1) and (4.2)

$$\frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}} \int_0^x y[(F(x))^{m+1} - (F(y))^{m+1}]^{s-r-1}$$

$$\times [F(y)]^{\gamma_s-1} f(y)dy = [F(x)]^{\gamma_{r+1}} H_r(x) , \qquad (4.7)$$

where

$$H_r(x) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^{p+q+(1/\beta)}(1-\alpha)^q (p/\lambda)\left[(1+x^{\beta})^{\lambda}-1\right]^{q+l}}{p!q!l!\left[(1+x^{\beta})^{\lambda}-(1-\alpha)\right]}$$

$$\times \frac{\Gamma\left(\frac{1}{\beta}+1\right)\Gamma\left(\frac{p}{\lambda}+l\right)}{\Gamma\left(\frac{1}{\beta}+1-p\right)\Gamma\left(\frac{p}{\lambda}+1-q\right)} \prod_{j=1}^{s-r}\left(\frac{r+j}{\gamma_{r+j}}+q+l\right)$$

Differentiating (4.7) both sides with respect to x, we get

$$\frac{C_{s-1}[F(x)]^m f(x)}{(s-r-2)!C_{r-1}(m+1)^{s-r-2}} \int_0^x y[(F(x))^{m+1} - (F(y))^{m+1}]^{s-r-2} [F(y)]^{\gamma_s-1} f(y)dy$$

$$= H_r'(x)[F(x)]^{\gamma_{r+1}} + \gamma_{r+1} H_r(x)[F(x)]^{\gamma_{r+1}-1} f(x)$$

Therefore,

$$\frac{\alpha\,\beta\lambda\, x^{\beta-1}(1+x^{\beta})^{\lambda-1}}{[(1+x^{\beta})^{\lambda}-(1-\alpha)][(1+x^{\beta})^{\lambda}-1]}$$

which proves that

$$F(x)=\frac{1-(1+x^{\beta})^{-\lambda}}{1-(1-\alpha)(1+x^{\beta})^{-\lambda}},$$

$x>0,\ \alpha\ ,\ \beta,\lambda>0.$

5. Application

Moments of order statistics play an important role in quality control testing and reliability to predict the failure of future items based on the times of few early failures. In this Section we suggest some application based on moments discussed in Section 2.

i) Estimation: The moments of order statistics and record values given in Section 2 can be used to obtain the best linear unbiased estimate of the parameters of the Marshall-Olkin Burr type XII distribution.

ii) Characterization: The Marshall-Olkin Burr type XII distribution given in (1.5) can be characterized by using conditional expectation of generalized order statistics as Theorem 4.1.

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