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THE INVERSE WEIBULL GENERATOR OF DISTRIBUTIONS: PROPERTIES AND APPLICATIONS

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ABSTRACT

In this work, we present the inverse Weibull generated (IW-G) family, a novel family of univariate distributions with two additional positive parameters derived from inverse Weibull random variables. In addition to offering several new models, the new family includes two additional families as special situations. For the new family, we investigate four unique models. The quantile function, ordinary and incomplete moments, probability weighted moments, Révnyi entropy, and order statistics are among the mathematical characteristics of the new family that are developed. The maximum likelihood approach is used to estimate the model parameters. Applications demonstrate that compared to a number of current lifespan models, the new family of distributions may provide a superior match.

Keywords: Inverse Weibull distribution; Moments; quantiles; Inverse Weibull-G family.

Introduction

An essential probability distribution that may be used to examine life-time data with certain monotone failure rates is the inverse Weibull (IW) distribution. According to Keller and Kamath (1982), it is an appropriate model to represent mechanical component deterioration events. Nelson (1982) asserts that the IW distribution fits a number of data sets well, including the timeframes to breakdown of an insulating fluid under continuous stress. The IW distribution has been the subject of several studies; for instance, parameter estimation of the distribution was addressed by Calabria and Pulcini (1990, 1994). Jiang et al. (2001) have presented several helpful metrics for the inverse Weibull distribution. Order statistics were obtained using the inverse Weibull distribution by Mahmoud et al. (2003). Sultan (2008) produced the estimators of the reliability and hazard functions for the unknown parameters of the inverse Weibull distribution by deriving the Bayesian estimators based on lower record values. Kundu and Howlader (2010) examined the Bayes estimates of the unknown parameters of the IW distribution under a squared error loss function using Type-II censored data. For the model parameters and acceleration factor for the inverse Weibull model, Hassan and Al-Thobety (2012) offered the best simple failure step stress

partly accelerated life tests. The constant-stress partly accelerated life test for the inverse Weibull model based on multiple censored data was covered by Hassan et al. (2015).

If the random variable X's cumulative distribution function (cdf) has the following form, it has an inverse Weibull distribution.

$$F(t) = e^{-\theta^{\beta} t^{-\beta}} \;\; ; \;\; t, \theta, \beta > 0, \qquad (1)$$

where, θ and β are the scale and shape parameters respectively. The corresponding probability density function (pdf) is given by

$$f(t) = \beta \theta^{\beta} \; t^{-\beta-1} \; e^{-\theta^{\beta} t^{-\beta}} \;\; ; \;\; t, \theta, \beta > 0 \qquad (2)$$

In recent years, some endeavors have been made to define new generators of continuous distributions from classic ones to provide great flexibility in modelling data in several applied areas. Several generated families have been suggested by several authors; see for examples, beta-G by Eugene et al. (2002), gamma-G by Zografos and Balakrishanan (2009), Kumaraswamy-G by Cordeiro and de Castro (2011), generalized beta-G by Alexander et al. (2012), exponentiated generalized (exp-G) by Cordeiro et al. (2013), transformed-transformer (T-X) by Alzaatreh et al. (2013), Weibull-G by Bourguignon et al. (2014), Kumaraswamy Weibull-G by Hassan and Elgarhy (2016 a), exponentiated Weibull-G by Hassan and Elgarhy (2016 b), additive Weibull-G by Hassan and Hemeda (2016), Type II half logistic by Hassan et al. (2017 a), generalized additive Weibull-G by Hassan et al. (2017 b), and power Lindley-G by Hassan and Nassr (2018) among others.

Alzaatreh et al. (2013), defined the T-X family of distributions as follows

$$F(x) = \int_{0}^{W[G(x)]} r(t) \; dt \qquad (3)$$

where, $r(t)$ is the pdf of a random variables T and $W[G(x)]$ be a function of the cdf of any random variables X.

In this article, we provide a new family of distributions using inverse Weibull as a generator with the hope that it will attract a wider application in some areas. This paper can be organized as follows. In Section 2, we provide a formation of the IW-G family. Four special models of IW-G family are defined in Section 3. Some useful expansions for the pdf and cdf of IW-G family are derived in Section 4. In the same section, explicit expressions for the moments, probability weighted moments, order statistics and quantile function are obtained. Estimation of the model parameters using maximum likelihood method is performed in Section 5. Section 6 provides two applications to two real data sets are presented to illustrate the potentially of the new family. Section 7 ends with some concluding remarks.

1. Inverse Weibull-G Family

In this section, we display the formation of the IW-G family of probability distributions. The cdf, reliability function, hazard rate function and cumulative hazard rate are derived.

Based on T-X generator, the distribution function of IW-G family is derived by replacing the generator, $r(t)$, in (3) by inverse Weibull generator defined in (1), with $W[G(x)] = G(x;\xi)/\bar{G}(x;\xi)$ as the following

$$F(x) = \int_0^{\frac{G(x;\xi)}{\bar{G}(x;\xi)}} \beta\theta^{\beta} t^{-\beta-1} e^{-\theta^{\beta} t^{-\beta}} dt = e^{-\theta^{\beta}\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{-\beta}} \quad (4)$$

where, $\bar{G}(x;\xi) = 1 - G(x;\xi)$. The associated pdf of the IW-G family can be written as follows

$$f(x) = \frac{\beta\theta^{\beta} g(x;\xi)[G(x;\xi)]^{-\beta-1}}{[G(x;\xi)]^{-\beta+1}} e^{-\theta^{\beta}\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{-\beta}} \quad ;\ x,\ \theta,\ \beta\ >\ 0 \quad (5)$$

A random variable X having IW-G density function (5) is denoted by X~IW-G.

For $\beta = 1$ IW-G reduces to new generated family called the inverse exponential-G family, for $\beta = 2$ the IW-G reduces to another generated family named the inverse Rayleigh-G family. Also, for $\theta = 1$, the IW-G reduces to the odd Frèchet-G family presented by Haq and Elgarhy (2018).

Furthermore, the reliability function; $\bar{F}(x)$, hazard rate function; $h(x)$ and cumulative hazard rate function $H(x)$ are, respectively, given by

$$\bar{F}(x) = 1 - e^{-\theta^{\beta}\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{-\beta}},$$

$$h(x) = \frac{\beta\theta^{\beta} g(x;\xi)[G(x;\xi)]^{-\beta-1}}{[G(x;\xi)]^{-\beta+1}} e^{-\theta^{\beta}\left[\frac{G(x;\xi)}{G(x;\xi)}\right]^{-\beta}} \left[1 - e^{-\theta^{\beta}\left[\frac{G(x;\xi)}{G(x;\xi)}\right]^{-\beta}}\right]^{-1},$$

$$H(x) = -\ln[F(x)] = -\ln\{1 - \exp(-\theta^{\beta}\left[\frac{G(x;\xi)}{G(x;\xi)}\right]^{-\beta})\}.$$

2. Special Models

A number of new distributions can be deduced as special models from the IW-G family of distributions. Here, four special models, namely; the inverse Weibull Weibull (IWW); the inverse Weibull Pareto (IWP); the inverse Weibull uniform (IWU) and the inverse Weibull Burr XII (IWXII) are introduced.

2.1. Inverse Weibull Weibull Model

Suppose, the Weibull cdf and pdf with parameters $\alpha > 0, \lambda > 0$ are $G(x) = 1 - e^{-(\lambda x)^{\alpha}}; x > 0$ and $g(x) = \alpha\lambda^{\alpha}x^{\alpha-1}e^{-(\lambda x)^{\alpha}}$. The cdf of a random variable X has the IWW distribution, say $X \sim IWW(\theta, \beta, \alpha, \lambda)$ is given by

$$f_{IWW}(x) = \alpha\beta\theta^{\beta}\lambda^{\alpha}x^{\alpha-1}e^{(\lambda x)^{\alpha}}(e^{(\lambda x)^{\alpha}} - 1)^{-\beta-1}\exp[-\theta^{\beta}(e^{(\lambda x)^{\alpha}} - 1)^{-\beta}]; x, \theta, \beta, \alpha, \lambda > 0 \quad (6)$$

The corresponding distribution function is given by

$$F_{IWW}(x) = \exp[-\theta^{\beta}(e^{(\lambda x)^{\alpha}} - 1)^{-\beta}].$$

The density function (6) reduces to inverse Weibull exponential distribution for $\alpha = 1$. Also, for $\alpha = 2$, the density function (6) reduces to inverse Weibul Rayleigh distribution. The reliability function and the hazard rate function (hrf) are obtained as follows

$$\bar{F}_{IWW}(x) = 1 - \exp[-\theta^{\beta}(e^{(\lambda x)^{\alpha}} - 1)^{-\beta}].$$

and,

$$h_{IWW}(x) = \alpha\beta\theta^{\beta}\lambda^{\alpha}x^{\alpha-1}e^{(\lambda x)^{\alpha}}(e^{(\lambda x)^{\alpha}} - 1)^{-\beta-1}e^{[-\theta^{\beta}(e^{(\lambda x)^{\alpha}}-1)^{-\beta}]}[1 - e^{(-\theta^{\beta}(e^{(\lambda x)^{\alpha}}-1)^{-\beta})}]^{-1}.$$

2.2. Inverse Weibull Pareto Model

The second model is the inverse Weibull Pareto whose cdf is derived by substituting the following cdf $G(x) = 1 - (\frac{x}{\lambda})^{-\alpha}$; $x > \lambda$ in (4) as follows

$$F_{IWP}(x) = \exp\left[-\theta^{\beta}\left(\left(\frac{x}{\lambda}\right)^{\alpha} - 1\right)^{-\beta}\right];\ \lambda, \alpha, \theta, \beta > 0,\ \lambda < x < \infty.$$

The probability density function of a random variable X having the IWP distribution, say $X \sim IWP(\theta, \beta, \alpha, \lambda)$ is given by

$$f_{IWP}(x) = \alpha\beta\theta^{\beta}\lambda^{-\alpha}x^{\alpha-1}\left(\left(\frac{x}{\lambda}\right)^{\alpha} - 1\right)^{-\beta-1}\exp\left[-\theta^{\beta}\left(\left(\frac{x}{\lambda}\right)^{\alpha} - 1\right)^{-\beta}\right].$$

Furthermore, the reliability function and hrf are as follows

$$\bar{F}_{IWP}(x) = 1 - \exp\left[-\theta^{\beta}\left(\left(\frac{x}{\lambda}\right)^{\alpha} - 1\right)^{-\beta}\right],$$

and,

$$h_{IWP}(x) = \alpha\beta\theta^{\beta}\lambda^{-\alpha}x^{\alpha-1}\left(\left(\frac{x}{\lambda}\right)^{\alpha} - 1\right)^{-\beta-1}e^{-\theta^{\beta}\left(\left(\frac{x}{\lambda}\right)^{\alpha}-1\right)^{-\beta}}\left[1 - e^{-\theta^{\beta}\left(\left(\frac{x}{\lambda}\right)^{\alpha}-1\right)^{-\beta}}\right]^{-1}.$$

2.3. Inverse Weibull Uniform Model

Considering the baseline distribution is uniform on the interval $(0, \alpha), \alpha > 0$, the cdf of inverse Weibull uniform distribution is as follows

$$F_{IWU}(x) = \exp\left[-\theta_{\beta}\left(\frac{x}{\alpha - x}\right)^{-\beta}\right];\ \alpha, \theta, \beta > 0,\ 0 < x < \alpha.$$

The probability density function of a random variable X having the IWU distribution, say $X \sim IWU(\theta, \beta, \alpha)$ is given by

$$f_{IWU}(x) = \alpha\beta\theta^{\beta}x^{-(\beta+1)}(\alpha - x)^{\beta-1}\left(\frac{x}{\alpha - x}\right)^{\beta}\exp\left[-\theta^{\beta}\left(\frac{x}{\alpha - x}\right)^{-\beta}\right].$$

Furthermore, the reliability and hazard rate functions are as follows

$$F_{IWU}(x) = 1 - \exp\left[-\theta_{\beta}\left(\frac{x}{\alpha - x}\right)^{-\beta}\right],$$

and,

$$h_{IWU}(x) = \alpha\beta\theta^{\beta}x^{-(\beta+1)}(\alpha - x)^{\beta-1}e^{-\theta}$$

=

) $_{\alpha-x}$

$\frac{\beta}{\alpha-x}(\frac{x}{\alpha-x})^{-\beta}$ $[1-e^{-\theta}$ $\beta(\frac{x}{\alpha-x})^{-\beta}$ $^{-1}$

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2.4. Inverse Weibull Burr XII Model

Zimmer et al. (1998) introduced the three-parameter Burr XII (BXII) distribution with the following pdf and cdf

$$g(x) = C\alpha\lambda^{-C}x^{C-1}\left[1+\left(\frac{x}{\lambda}\right)^{C}\right]^{-\alpha-1}; x>0, C, \alpha, \lambda > 0,$$

$$G(x) = 1-\left[1+\left(\frac{x}{\lambda}\right)^{C}\right]^{-\alpha}.$$

Hence, the pdf of a random variable X has the inverse Weibull Burr XII distribution, say $X \sim IWBXII(\theta, \beta, \alpha, \lambda, C)$ is obtained from cdf (4) as follows

$$F_{IWBXII}(x) = \exp\left[-\theta^{\beta}\left\{\left(1+\left(\frac{x}{\lambda}\right)^{C}\right)^{\alpha}-1\right\}^{-\beta}\right]; x>0, \alpha, \theta, \beta, \lambda, C > 0.$$

The pdf of the IWBXII is given by

$$f_{IWBXII}(x) = \alpha\beta C\theta^{\beta}\lambda^{-C}x^{C-1}\left(1+\left(\frac{x}{\lambda}\right)^{C}\right)\left\{\left(1+\left(\frac{x}{\lambda}\right)^{C}\right)^{\alpha}-1\right\}^{-\beta-1} e^{-\theta^{\beta}\left\{\left(1+\left(\frac{x}{\lambda}\right)^{C}\right)^{\alpha}-1\right\}^{-\beta}}.$$

The reliability and hazard rate functions are

$$F_{IWBXII}(x) = 1-\exp\left[-\theta^{\beta}\left\{\left(1+\left(\frac{x}{\lambda}\right)^{C}\right)^{\alpha}-1\right\}^{-\beta}\right],$$

$$h_{IWBXII}(x) = \alpha\beta C\theta^{\beta}\lambda^{-C}x^{C-1}\left(1+\left(\frac{x}{\lambda}\right)^{C}\right)^{\alpha-1}\left\{\left(1+\left(\frac{x}{\lambda}\right)^{C}\right)^{\alpha}-1\right\}^{-\beta-1} e^{-\theta^{\beta}\left\{\left(1+\left(\frac{x}{\lambda}\right)^{C}\right)^{\alpha}-1\right\}^{-\beta}}\left[1-e^{-\theta^{\beta}\left\{\left(1+\left(\frac{x}{\lambda}\right)^{C}\right)^{\alpha}-1\right\}^{-\beta}}\right]^{-1}.$$

In Figure 1, we display some plots of the pdf of the IWW, IWP, IWU and IWBXII for selected parameter values. Figure1 reveals that the IWW, IWP, IWU and IWBXII densities generate various shapes such as symmetrical, left-skewed, reversed-J, unimodal and U shaped.

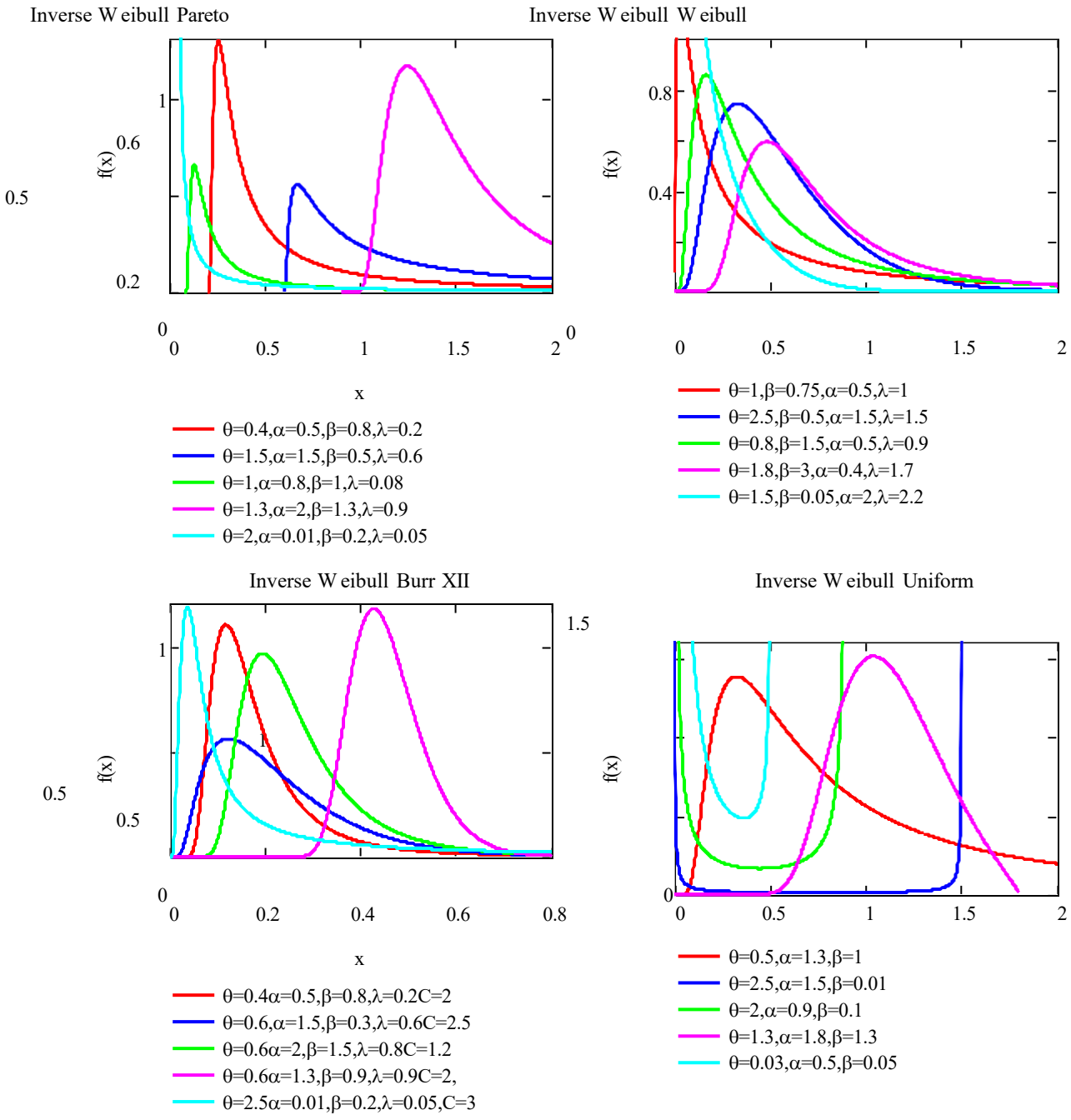


Figure 1: Density plots of the *IWW, IWP, IWU* and *IWBXII* models

Plots of the hrf of the IWW, IWP, IWU and IWBXII models are described in Figure 2 for some selected parameter values. From Figure 2, we observe that these models can

produce hazard rate shapes such as constant, increasing, decreasing, and upside-down bathtub. This fact implies that the IW-G family can be very useful for fitting data sets with various shapes.

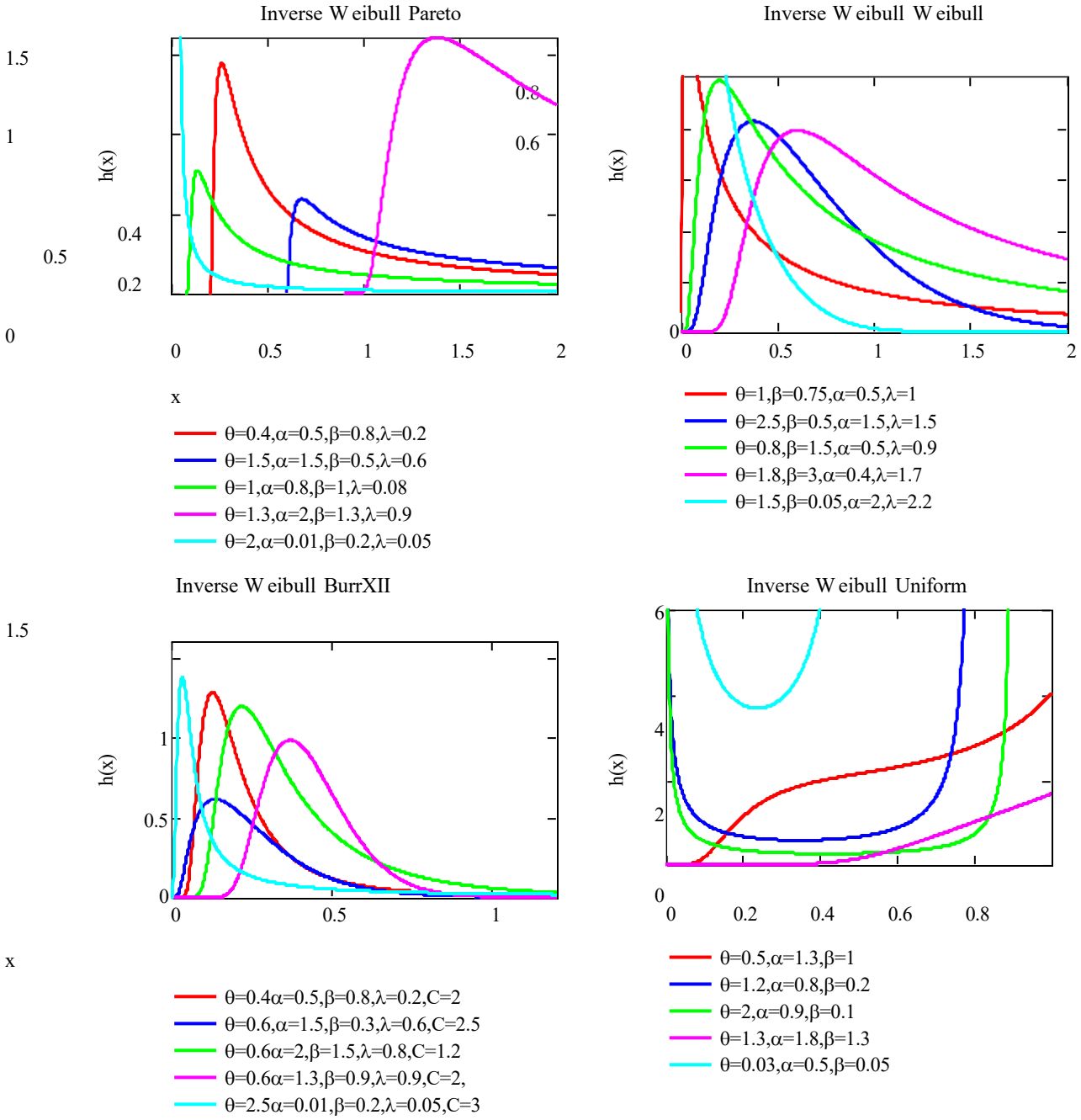


Figure 2: Hazard rate function plots of the *IWW*, *IWP, IWU* and *IWBXII* models

3. Mathematical Properties

In this section, we provide some main mathematical properties of the IW-G family.

3.1. Useful Expansions

Some mathematical properties of the IW-G family can be confirmed through an algebraic expansion which is more efficient than computing those directly by numerical integration of its density function. Here two important expansions are deduced for the IW-G pdf and cdf using mixture forms of exponentiated-G (Exp-G) distribution.

Since, the power series for the exponential function in pdf (5) can be written as follows:

$$e^{-\theta\beta\left(\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right)^{-\beta}} = \sum_{i=0}^{\infty} \frac{(-1)^i\,\theta^{i\beta}}{i!}\left(\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right)^{-\beta i} \tag{7}$$

Inserting expansion (7) in pdf (5), then we have

$$f(x) = \beta \sum_{i=0}^{\infty} \frac{(-1)^i\,\theta^{\beta(i+1)} g(x;\xi)(G(x;\xi))^{-(\beta i+\beta+1)}}{i!\;(\bar{G}(x;\xi))^{-(\beta i+\beta+1)}} \tag{8}$$

It is well-known that, if $\beta > 0$ is real non integer and $|z| < 1$, the generalized binomial theorem is written as follows

$$(1-z)^{\beta-1} = \sum_{j=0}^{\infty} (-1)^j \binom{\beta-1}{j} z^j \tag{9}$$

Then, by applying the binomial theorem (9) in (8), the IW-G pdf, where β is real non integer becomes

$$f(x) = \sum_{i,j=0}^{\infty} \binom{\beta i+\beta-1}{j} \frac{(-1)^{i+j}\beta\,\theta^{\beta(i+1)}}{i!} g(x;\xi)(G(x;\xi))^{j-\beta(i+1)-1}.$$

Hence, IW-G density function can be expressed as an infinite mixture of Exp-G density functions as follows

$$f(x) = \sum_{i,j=0}^{\infty} \eta_{i,j}\, W_k(x;\xi)\,; \quad x > 0, \tag{10}$$

$$\eta_{i,j} = \binom{\beta i+\beta-1}{j} \frac{(-1)^{i+j}\beta\,\theta^{\beta(i+1)}}{i!\;k}\,, k = j-\beta(i+1) \Big\}$$

where, $W_k(x;\xi) = k g(x;\xi)(G(x;\xi))^{k-1}$ denotes the 'exp-G' pdf with power parameter k.

Additionally, using the exponential expansion for $[F(x)]^s$, where $s$is a positive integer, becomes

$$[F(x)]^s = \sum_{m=0}^{\infty} \frac{(-1)^m\, s^m\, \theta^{\beta m}}{m!} \left(\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right)^{-\beta m} \tag{11}$$

Using the binomial expansion (9) in (11) where β is a real non integer, leads to :

$$[F(x)]^s = \sum_{m,l=0}^{\infty} \psi_{m,l}\; (G(x;\xi))^{l-\beta m} \tag{12}$$

where, $\psi_{m,l} = \frac{(-1)^{m+l}\, s^m\, \theta^{\beta m}}{m!}\binom{\beta m+1}{l}$.

3.2. Quantile Function

In this subsection, the quantile function of a random variable X has the IW-G distribution is derived. More specifically, the quantile function for the IWP model is obtained. Further, the skewness and kurtosis based on quantile function for IWP model are discussed.

The quantile function of IW-G family, say $x = Q(x) = F^{-1}(u)$ can be obtained by inverting (4) as follows

$$x = Q(x) = F^{-1}(u) = G^{-1}\{(-\theta^{-\beta}\ln(u))^{-1/\beta}\,[1 + (-\theta^{-\beta}\ln(u))^{-1/\beta}]\}.$$

Solving this equation for $Q(x)$ gives the quantile function.

Example 1: Consider the IWP distribution discussed in subsection (3.2). The quantile function of IWP is obtained as follows

$$x = Q(x) = \lambda[1 + \theta(-\ln(u))^{-1/\beta}]^{1/\alpha}.$$

Kenney and Keeping (1962) proposed skewness based on quartiles called the Bowley skewness which is defined as follows

$$B = \frac{Q(\frac{3}{4})-2Q(\frac{1}{2})+Q(\frac{1}{4})}{Q(\frac{3}{4})-Q(\frac{1}{4})}.$$

Further, the Moors kurtosis (see Moors (1988)) based on octiles is defined as follows

$$M = \frac{Q(\frac{7}{8})-Q(\frac{5}{8})+Q(\frac{3}{8})-Q(\frac{1}{8})}{Q(\frac{6}{8})-Q(\frac{2}{8})}.$$

where, $Q(.)$ denotes the quantile function. Figure 3 gives plots of the skewness and kurtosis for some choices of the parameter β as function of α. These plots indicate that the skewness and kurtosis decrease when β increases for fixed α.

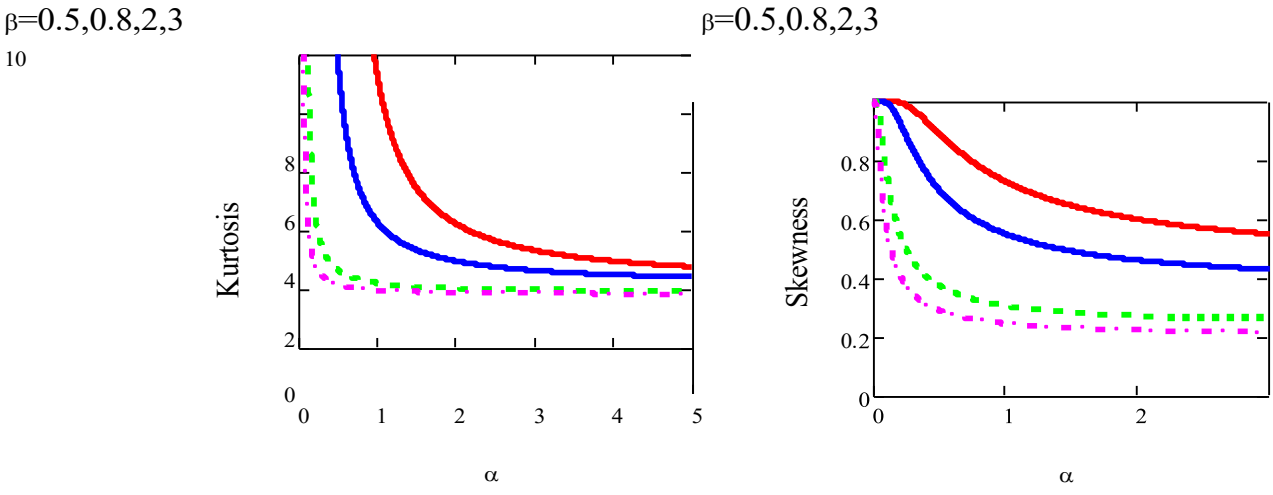


Figure 3: Plots of skewness and kurtosis for *IWP* distribution based on quantile.

3.3. Moments

Most of the necessary characteristics and features of a distribution can be studied through its moments. Here, the moments of a random variable X has the IW-G are derived. More specifically, the rth moment for the IWU model is obtained.

The rth moment of X about the origin is derived from (10) as follows

$$\acute{\mu}_r = \sum_{i,j=0}^{\infty} \eta_{i,j} \int_{-\infty}^{\infty} x^r W_k(x;\xi)\, dx \ , \quad r = 1, 2, \ldots \tag{13}$$

Furthermore, the moment generating function of X is

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \acute{\mu}_r = \sum_{r=0}^{\infty} \sum_{i,j=0}^{\infty} \frac{t^r}{r!} \eta_{i,j} \int_{-\infty}^{\infty} x^r W_k(x;\xi)\, dx .$$

Example 2: Consider the IWU distribution discussed in subsection (3.3). The rth moment of IWU can be obtained from (13) with pdf and cdf as defined in subsection (3.3) as follows

$$\acute{\mu}_r = \sum_{i,j=0}^{\infty} \eta_{i,j} \frac{j-\beta(i+1)}{r+(j-\beta(i+1))} \alpha^r.$$

In particular, the mean and variance of the IWU distribution are obtained, respectively, as follows:

$$E(X) = \sum_{i,j=0}^{\infty} \eta_{i,j} \frac{j-\beta(i+1)}{(j-\beta(i+1))+1} \alpha,$$

$$Var(X) = \sum_{i,j=0}^{\infty} \eta_{i,j} \frac{j-\beta(i+1)}{(j-\beta(i+1))+2} \alpha^2 - \left[\sum_{i,j=0}^{\infty} \eta_{i,j} \frac{j-\beta(i+1)}{(j-\beta(i+1))+1} \alpha\right]^2 .$$

3.4. The probability Weighted Moments

The probability-weighted moments (PWMs) method of estimation has been proposed by Greenwood et al. (1979) for distribution expressible in inverse form. For a random variable X the PWMs, denoted by $\tau_{r,s}$, can be calculated through the following relation

$$\tau_{r,s} = E[X^r F(x)^s] = \int_{-\infty}^{\infty} x^r f(x) [F(x)]^s \, dx \tag{14}$$

The PWMs of IW-G family is obtained by inserting (10) and (12) into (14) as follows

$$\tau_{r,s} = \sum_{i,j=0}^{\infty} \sum_{m,l=0}^{\infty} \zeta_{i,j,k,m,l} \int_{-\infty}^{\infty} x^r \, g(x;\xi) \, [G(x;\xi)]^{l-\beta m+k-1} \, dx,$$

where, $\zeta_{i,j,k,m,l} = \psi_{m,l}\, \eta_{i,j}$ k and $k = j - \beta(i+1)$.

Example 3: Consider the IWP distribution discussed in subsection (3.2). The PWMs of IWP distribution can be obtained as follows

$$\int_{-\infty}^{\infty} x^r \alpha x^{-\alpha-1} x^{-\alpha(l-\beta m+k-1)}$$

$$\tau_{r,s} = \sum_{i,j,m,l=0} \zeta_{i,j,k,m,l} \int_{\lambda} x \quad \frac{}{\lambda} \left(\frac{}{\lambda}\right)_{\lambda} \quad [1 - (-) \quad] \quad dx ,$$

$$\tau_{r,s} = \sum_{i,j,m,l=0}^{\infty} \zeta_{i,j,k,m,l} \; \lambda^{r} \; B\left(1 - \frac{r}{\alpha}, l - \beta m + k\right).$$

where, $B(.,.)$ stands for beta function and $k = j - \beta(i + 1)$.

3.5. The Mean Deviation

In statistics, the mean deviation about the mean and mean deviation about the median measure the amount of scattering in a population. For random variable X with pdf $f(x)$, cdf $F(x)$, the mean deviation about the mean and mean deviation about the median, are defined by

$$\delta_1(X) = 2\mu F(\mu) - 2T(\mu) \text{ and } \delta_2(X) = \mu - 2T(M)$$

where, $\mu = E(X)$, $M = \text{Median}\ (X)$, and $T(q) = \int_{-\infty}^{q} x\, f(x)\ dx$ which is the first incomplete moment. Another application of the first incomplete moment refers to the Bonferroni and Lorenz curves. These curves are very useful in economics, reliability, demography, insurance and medicine.

Example 4: Consider the IWU distribution discussed in subsection (3.2). The first incomplete moment of IWU distribution can be obtained as follows

$$T(q) = \sum_{i,j=0}^{\infty} \eta_{i,j} \int_0^q x^k \left(\frac{\alpha}{x}\right)^{\alpha} dx = \sum_{i,j=0}^{\infty} \eta_{i,j} \frac{q^{k+1}}{k+1},$$

where, $\eta_{i,j} = \binom{\beta i + \beta - 1}{j} \frac{(-1)^{i+j}\beta\theta^{\beta(i+1)}}{i!\ k}$ and $k = j - \beta(i+1)$.

3.6. Order Statistics

Suppose $X_1, X_2, \ldots, X_n$ be independent and identically distributed (i.i.d) random variables with their corresponding continuous distribution function $F(x)$. Let $X_{1:n} < X_{2:n} < \ldots < X_{n:n}$ the corresponding ordered random sample from a population of size n. According to David (1981), the pdf of the rth order statistic, is defined by

$$f_{r:n}(x) = \frac{1}{B(r, n-r+1)} \sum_{v=0}^{n-r} (-1)^r \binom{n-r}{v} f(x) F(x)^{v+r-1} \tag{15}$$

The pdf of the rth order statistic for IW-G family is derived by substituting (10) and (12) in (15), replacing s with $v + r - 1$,

$$f_{r:n}(x) = \frac{1}{B(r,n-r+1)} \sum_{v=0}^{n-r} \sum_{i,j,m,l=0}^{\infty} (-1)^r \binom{n-r}{v} \eta_{i,j} \psi_{m,l}\, g(x;\xi)[G(x;\xi)]^{l-\beta m+j-\beta(i+1)-1}$$

$$f_{r:n}(x) = \sum_{v=0}^{n-r} \sum_{i,j,m,l=0}^{\infty} N_{v,i,j,m,l}\, W_{k_1}^{*}(x;\xi),$$

$$N_{v,i,j,m,l} = \frac{k\theta^{\beta m}(v+r-1)^m(-1)^{r+m+l}}{B(r,n-r+1)k_1\, m!} \binom{n-r}{v} \binom{\beta m+1}{l} \eta_{i,j},\ k_1 = l - \beta m + j - \beta(i+1) \tag{16}$$

where, $W_{k_1}^{*}(x;\xi)$ denotes the 'exp-G' pdf with power parameter k_1. In particular, the pdf of the smallest order statistics $X_{1:n}$ is obtained from (16), by substituting $r = 1$, as follows

$$f_{1:n}(x) = \sum_{v=0} \sum_{i,j,m,l=0} M_{v,i,j,m,l} \, W^*_{k_1}(x;\xi) \,,$$

where, $M_{v,i,j,m,l} = \frac{nk(-1)^{m+l+1} \theta^{\beta m} v^m}{k_1 \, m!} \binom{n-1}{v} \binom{\beta m + 1}{l} \eta_{i,j}$.

Also, the pdf of the largest order statistics $X_{n:n}$ is obtained by substituting $r = n$ in (16),

$$f_{n:n}(x) = \sum_{i,j,m,l=0}^{\infty} L_{i,j,m,l}\, W^{*}_{k_1}(x;\xi)\,,$$

where, $L_{i,j,m,l} = \dfrac{nk\theta^{\beta m}(-1)^{n+m+l}\binom{v+n-1}{m}\binom{\beta m+1}{l}}{k_1\, m!}\,\eta_{i,j}$.

3.7. Rènyi Entropy

Entropy is a measure of variation or uncertainty of a random variable X (Rènyi, 1961). The Rènyi entropy of a random variable is defined by

$$I_{\delta}(X) = \frac{1}{1-\delta}\log\int_{-\infty}^{\infty} f(x)^{\delta}\,dx\,,\ \ \delta > 0 \text{ and } \delta \neq 1 \tag{17}$$

By applying the binomial theory (9) and exponential expansion, then the pdf $f(x)^{\delta}$ can be expressed as follows

$$f(x)^{\delta} = \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}\,\delta^{i}\,\theta^{\beta(\delta+i)}}{i!}\binom{\delta(\beta+\beta i-1)+1}{j}(g(x;\xi))^{\delta}(G(x;\xi))^{j-\delta(\beta+\beta i+1)} \tag{18}$$

Inserting (18) in (17), then the Rènyi entropy of the family is as follows

$$I_{\delta}(X) = \frac{1}{1-\delta}\log\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}\,\delta^{i}\,\theta^{\beta(\delta+i)}}{i!}\binom{\delta(\beta+\beta i-1)+1}{j}\Im(\delta,i,j)\,,$$

where, $\Im(\delta,i,j) = \int_{-\infty}^{\infty}(g(x;\xi))^{\delta}(G(x;\xi))^{j-\delta(\beta+\beta i+1)}\,dx.$

Example 5: Consider again, the IWP distribution discussed in subsection (3.2). The Rènyi entropy of IWP distribution can be derived as follows

$$I_{\delta}(X) = \frac{1}{1-\delta}\log\left[\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}\,\delta^{i}\,\theta^{\beta(\delta+i)}}{i!}\binom{\delta(\beta+\beta i-1)}{j}\left(\frac{\alpha}{\lambda}\right)^{\delta-1} B\left(\frac{\delta(\alpha+1)-1}{\alpha}, j-\right.\right.$$

$\left.\left.\delta(\beta+\beta i-1)+1\right)\right]$.

4. Estimation of Parameters

This section concerns with the maximum likelihood estimates (MLEs) of the unknown parameters for the new family based on complete samples. Let $X_1, X_2, \ldots, X_n$ be a simple

random sample from pdf (5) with set of parameters $\Theta \equiv (\beta, \theta, \xi)$. The log-likelihood function, denoted by $\ln L$, based on the observed random sample of size n from density (5) is given by:

$$\ln L = n \ln \beta + n\beta \ln \theta + \sum_{i=1}^{n} \ln g(x_i, \xi) - 2 \sum^{n} \ln G(x_i, \xi) -$$

$$(\beta + 1) \sum^{n} \ln[Z(x_i, \xi)] - \theta^{\beta} \sum^{n} [Z(x_i, \xi)]^{-\beta},$$

$i=1$ $i=1$

where, $Z(x_i, \xi) = \frac{G(x_i, \xi)}{\bar{G}(x_i, \xi)}$.

The elements of the score function $U(\Theta) = (U_\theta, U_\beta, U_\xi)$ are given by $U_\theta = \frac{n\beta}{\theta} - \beta\theta^{\beta-1}\sum_{i=1}^{n}[Z(x_i,\xi)]^{-\beta}$,

$$U_\beta = \frac{n}{\beta} + n\ln\theta - \sum_{i=1}^{n}\ln[Z(x_i,\xi)] + \theta^\beta \sum_{i=1}^{n}[Z(x_i,\xi)]^{-\beta}\ln[Z(x_i,\xi)] - \theta^\beta \ln\theta \sum_{i=1}^{n}[Z(x_i,\xi)]^{-\beta},$$

and

$$U_\xi = \sum_{i=1}^{n}\frac{\partial g(x_i,\xi)/\partial\xi_k}{g(x_i,\xi)} - 2\sum_{i=1}^{n}\frac{\partial G(x_i,\xi)/\partial\xi_k}{G(x_i,\xi)} - (\beta+1)\sum_{i=1}^{n}\frac{\partial Z(x_i,\xi)/\partial\xi_k}{Z(x_i,\xi)} + \beta\theta^\beta\sum_{i=1}^{n}[Z(x_i,\xi)]^{-\beta-1}\frac{\partial Z(x_i,\xi)}{\partial\xi_k}.$$

Setting U_θ, U_β and U_ξ equal to zero and solving the equations simultaneously the MLE, say $\hat{\Theta} \equiv (\beta, \theta, \xi)^T$ of $\Theta \equiv (\beta, \theta, \xi)^T$ are obtained. Further the resulting equations cannot be solved analytically, so some software's can be used to solve them numerically.

For interval estimation of the model parameters, we obtain the 3×3 observed information matrix $I = I(\Theta)$ (for $r, s = \beta, \theta, \xi$), whose elements are listed in Appendix. Under standard regularity conditions, the multivariate normal $N_3\,(0, I^{-1}(\hat{\Theta}))$ distribution is used to construct approximate confidence intervals for the parameters. The approximate $100(1-\gamma)\%$ two sided confidence intervals for θ, β, ξ are respectively, given by:

$$\hat{\alpha} \pm Z_{\gamma/2}\sqrt{Var(\hat{\alpha})}\ ,\ \beta \pm Z_{\gamma/2}\sqrt{Var(\beta)} \text{ and } \xi \pm Z_{\gamma/2}\sqrt{Var(\xi)}\ .$$

Here, $Z_{\gamma/2}$ is the upper $\gamma/2^{th}$ percentile of the standard normal distribution and $Var(.)$'s denote the diagonal elements of $I^{-1}(\hat{\Theta})$ corresponding to the model parameters.

5. Applications To Real Data

The flexibility of IWW as especial model from IW-G family is examined using two real data sets. The superiority of IWW is clarified as compared with some main four models; additive Weibull (AW; Almalki and Yuan (2013)), new modified Weibull (NMW; Doostmoradi et al. (2014)), Weibull Weibull (WW; Bourguignon et al. (2014)) and Weibull (W).

For both data sets, the unknown parameters of each distribution are estimated by the maximum-likelihood method. The model selection is carried out using Kolmogorov-Smirnov (K-S) statistic and corresponding P-value, -2 log-likelihood function ($-2\ln L$), Akaike information criterion (AIC), the correct Akaike information criterion (CAIC),

Bayesian information criterion (BIC) and Hannan-Quinn information criterion (HQIC). However, the better distribution corresponds to the smaller values of AIC, CAIC, BIC, HQIC, K-S criteria and largest values of P-value. Furthermore, we plot the histogram for each data set and the estimated pdf of the four models. Moreover, plots of empirical cdf of the data sets and estimated cdf of four models are displayed.

Example 6.1: The first data set is provided in Murthy et al. (2004) about time between failures for repairable item. The data are listed as the following:

1.43, 0.11, 0.71, 0.77, 2.63, 1.49, 3.46, 2.46, 0.59, 0.74, 1.23, 0.94, 4.36, 0.40, 1.74, 4.73, 2.23, 0.45, 0.70, 1.06, 1.46, 0.30, 1.82, 2.37, 0.63, 1.23, 1.24, 1.97, 1.86, 1.17.

In Table (1), we list the values of AIC, CAIC, BIC, HQIC, K-S and the P-value statistics. We observe that the IWW model has the smallest AIC, CAIC, BIC, HQIC, K-S values, and has the largest P-value as compared with those values of the other models. So, the IWW model seems to be a very competitive model to this data. More information is provided by a visual comparison of the histogram and estimated cumulative of the data with the fitted models as shown in Figure 4. It is clear from Figure 4 that the IWW distribution provides a better fit than the other competitive models.

Table (1): Model Selection Criteria for the First Data Set

Model	**-2lnL**	***AIC***	***BIC***	***CAIC***	***HQIC***	***K-S***	**P-value**
IWW	79.21	87.21	92.815	88.81	89.003	0.070	998.0
NMW	242.501	250.051	255.656	251.651	251.845	0.942	0.000
AW	159.642	167.642	173.246	169.242	169.435	0.283	0.017
WW	80.276	88.276	93.88	89.876	90.069	0.086	0.978
W	92.751	96.751	99.554	97.196	97.648	0.134	0.655

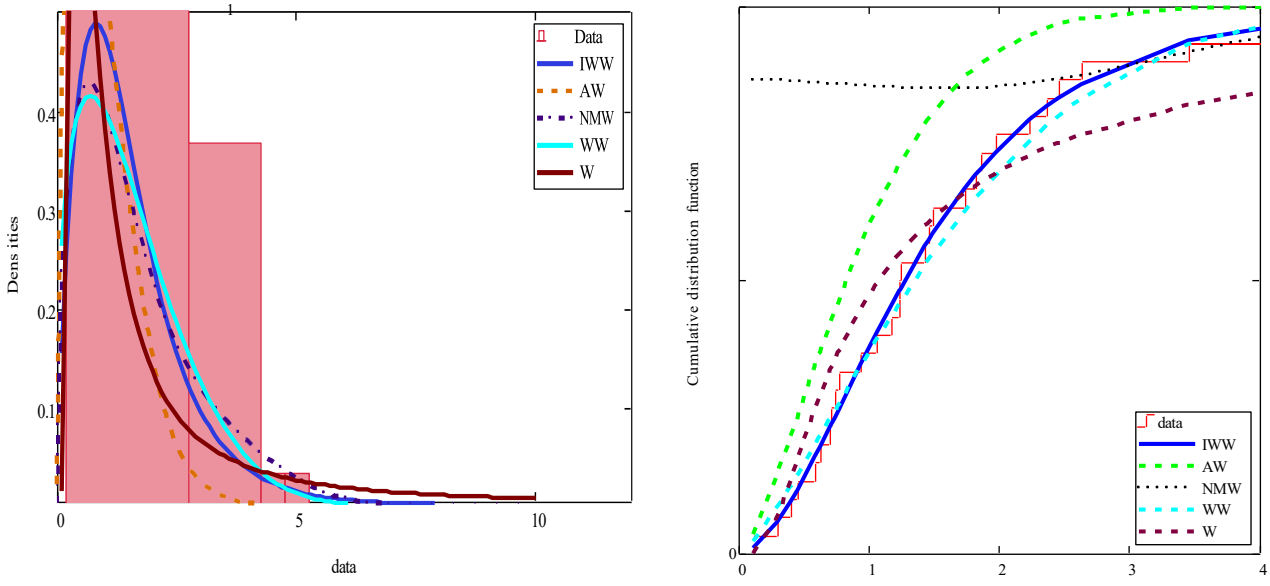


Figure 4: Estimated densities and estimated distributions of models for the first data set

Example 6.2: The second data set represents 34 observations of the vinyl chloride data obtained from clean up gradient ground –water monitoring wells in mg/L. The data are obtained from Bhaumik et al. (2009) and recorded as follows

5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1.0, 0.2, 0.1, 0.1, 1.8, 0.9, 2.0, 4.0, 6.8, 1.2, 0.4,0.2.

Results in Table 2, indicate that the IWW model is more suitable than the other competitive models for this data set based on the selected criteria. Further, it is clear from Figure 5 that the IWW distribution provides a better fit and therefore be one of the best models for this data set.

Table (2): Model Selection Criteria for the Second Data Set

Model	**-2lnL**	***AIC***	***BIC***	***CAIC***	***HQIC***	***K-S***	**P-value**
IWW	108.470	116.47	122.576	117.850	118.552	0.087	959.0
NMW	314.807	322.807	328.912	324.186	324.889	1	0.000
AW	221.798	229.798	235.904	231.178	231.881	0.982	0.018
WW	111.160	119.160	125.265	120.539	121.242	0.094	0.922
W	117.253	121.253	124.306	121.640	122.294	0.113	0.775

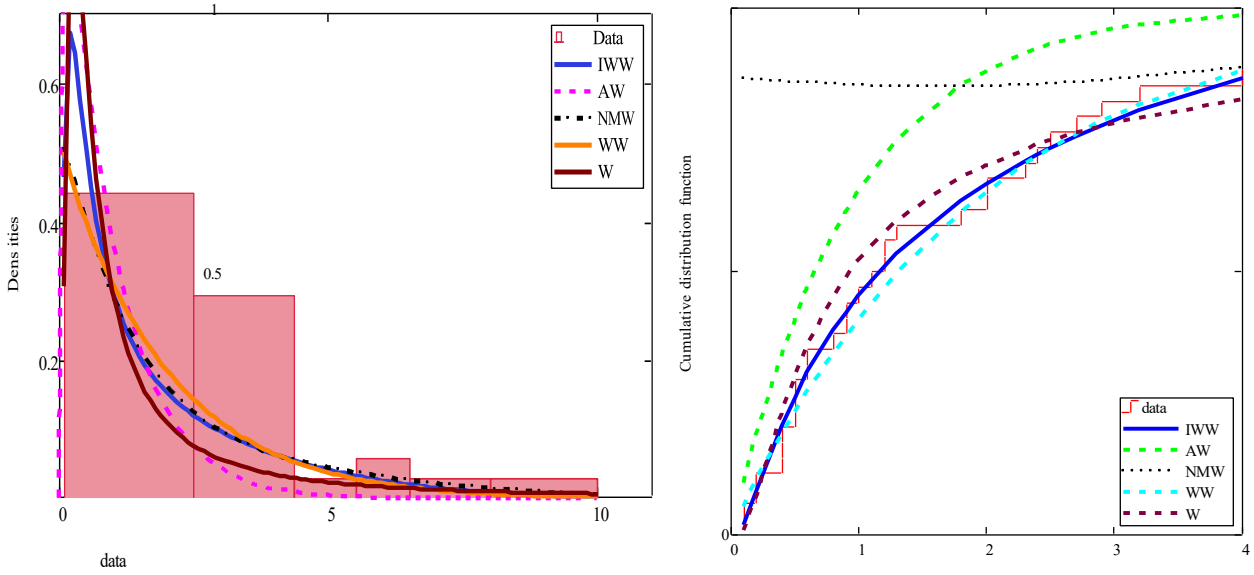


Figure 5: Estimated densities and estimated distributions of models for the second data set.

6. Conclusion

In this article, we provide a novel generator for a family of univariate distributions based on the inverse Weibull distribution. Four unique models are offered, along with several new sub-types. A combination of exponentiated-G distribution functions may be used to define the IW-G density function. The IW-G family's mathematical characteristics are obtained. For the moments, probability weighted, entropy, and distribution of order statistics, we provide precise closed form formulations. The model parameters and the observed Fisher information matrix are derived using the maximum likelihood technique of estimation. To demonstrate the versatility of this novel family, we fitted the IWW distribution as a special sub-model to two actual data sets. We anticipate that this next generation will find broader applicability across several domains.

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